Towards Intuitionistic N-Graphs

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Abstract. N-Graphs, introduced by De Oliveira in 2001, are a proof system whose derivations are represented by means of digraphs. These graphs are mostly based on Gentzen's natural deduction and sequent calculus formalisms, but in addition, the system combines four geometric approaches to deduction: tables of development (Kneale, 1957), proof-nets (Girard, 1987), logical flow graphs (Buss, 1991), and especially proofs-as-graphs (Statman, 1974).
In this paper, we introduce N-Graphs for intuitionistic logic. De Oliveira proposed and developed N-Graphs for classical propositional logic. Here we extend this graphical proof system to intuitionistic logic. We review the intuitionistic sequent calculus, in particular Gentzen's work, as well as two multiple conclusions versions for intuitionistic logic, namely the system *LJ'* (Maehara, 1954) and the system *FIL* (De Paiva and Pereira, 2005). We discuss problems and possible solutions for the construction of a proof system akin to N-Graphs but tailored to intuitionistic logic, thus arriving at our *intuitionistic N-Graphs*. We show soundness and completeness of the intuitionistic N-Graphs and discuss future developments.

Keywords: Proof Theory, Proof Graphs, N-Graphs, Intuitionistic Logic, Sequent calculus, Multiple-conclusion systems.

1 Introduction

The N-Graphs system is a multiple-conclusion proof system for classical propositional logic, developed by de Oliveira (more details can be found in [dO01], [dOdQ03]). The N-Graphs system has logical rules corresponding to natural deduction rules and also to some of the structural rules of the sequent calculus. A brief explanation of N-Graphs is presented in Section 2.

The multiple-conclusion system N-Graphs was essentially classical, that is, it did not have a version for intuitionistic propositional logic, as originally conceived. (There are also no intuitionistic versions of tables of development, proofs-as-graphs or logical flow graphs, as far as we know.) We propose a way of constructing a proof system akin to N-Graphs, but tailored to intuitionistic logic.

In Gentzen's original work [Gen35] the difference between the intuitionistic system, the calculus *LJ* and the classical system, the calculus *LK*, is the cardinality restriction of the succedent of the sequents. The formulation of the system *LK* uses sequent expressions of the form $\Gamma \vdash \Delta$. One such sequent is intuitively read as $A_1 \wedge ... \wedge A_n \rightarrow B_1 \vee ... \vee B_m$, the conjunction of the premisses entails the disjunction of the conclusions. In the calculus for intuitionistic logic, *LJ*, sequents are restricted to succedents with at most one formula occurrence, usually represented by $\Gamma \vdash B$. However, there are several well-known multiple-conclusion systems for intuitionistic logic.

First recall the system *LJ'* proposed by Maehara [Mae54], which is described in Takeuti's book [Tak75]. This system is obtained by setting restrictions on the calculus *LK* (instead of *LJ*) in the two rules of inference *negation on the right* ($R\neg$) and *implication on the right* ($R\rightarrow$), as presented below. All other rules of inference are the same as the rules of the *LK* system. (Note that since negation can be defined in terms of falsehood as $\neg A = A \rightarrow \bot$, only the modification on the implication rule is essential.)

$$\frac{D, \Gamma \vdash}{\Gamma \vdash \neg D}\ R\neg \qquad\qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \rightarrow B}\ R\rightarrow$$

Another multiple conclusion proof system for propositional intuitionistic logic was developed by de Paiva and Pereira, the system *FIL* [PP05]. This system uses a indexing device in the sequent, which allows tracking of the dependency relations between formulas in the antecedent and in the succedent of the sequent. A condition in the *right implication* rule ($R\rightarrow$) ensures that only valid constructive formulas are derived. The system *FIL* has the advantage of being closer to true multiple-conclusions as used in classical logic, but still retains some form of constructive constraint, otherwise it would be able to derive all theorems of classical logic.

In this work we choose the system *LJ'* to base our intuitionistic N-Graphs on. We first present a description of N-Graphs in section 2. Then we show a description of N-Graphs for intuitionistic logic based on *LJ'* in section 3. In section 4 we show how to prove the soundness and completeness for these intuitionistic N-Graphs. In section 5 we conclude with discussion of future work.

2 N-Graphs

In this section we introduce N-Graphs, the formal system developed by de Oliveira [dO01]. We begin with basic definitions of N-Graphs then we see how to build an N-Graph derivation and we describe the soundness criterion. Finally, we present two examples of derivations in the proof system.

2.1 Definitions

N-Graphs is a proof system with multiple-conclusion for the propositional classic calculus, and is based on logical rules of natural deduction and some of the structural rules of sequent calculus. N-Graphs are directed graphs that represent proofs where each vertex is an occurrence of a formula and each edge represents an atomic step in a derivation. In this formalism, proofs are represented by graphs which are constructed from a set of basic links illustrated in Fig. 2. Basic links represent schematically the rules of the calculus.

The propositional language uses logical constants: $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\neg$ (negation), the constant $\bot$ for *absurdity* (or *false*) and the constant $\top$ for *truth*. We use the letters $A, B, C, D, E, \ldots$ for arbitrary formulas (or formula-occurences) in the language.

First we need some definitions:

Definition 1 (Focussing/Defocussing branch point). *A* focussing branch point *is a vertex in a digraph with two edges oriented towards it. A* defocussing branch point *is a vertex in a digraph with two edges oriented away from it.*

Definition 2 (Focussing/Defocussing/Simple links). *A* focussing link *is a set $\{(u_1, v), (u_2, v)\}$ in a digraph in which v is a focussing branch point as illustrated in Fig. 1 The vertices u_1 and u_2 are called premisses of the link, while the vertex v is the conclusion. A* defocussing link *is a set $\{(u, v_1), (u, v_2)\}$ in a digraph in which u is a defocussing branch point as illustrated in Fig. 1 The vertices v_1 and v_2 are called conclusions of the link, while u is the premise. A* simple link *is an edge (u, v) in a digraph which neither belongs to a focussing nor to a defocussing link. The vertex u is called the premise of the link and v its conclusion.*

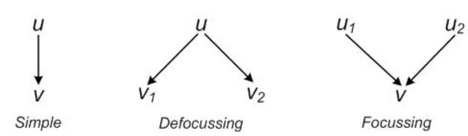


Fig. 1. Kinds of links

N-Graphs are somewhat similar to proof-nets. The same way we first define proof structures and then provide a criterion to determine which proof structures are logically sound, hence proof-nets, we first define proof-graphs, define a criterion to determine which proof-graphs are logically sound and then call these the N-graphs.

Definition 3 (Proof-graph). *A* proof-graph *is a connected oriented graph defined as follow:*

1. *each vertex is labelled with a formula-occurrence;*
2. *the edges are of two kinds ("meta" and "solid"), the meta-edges are labelled by a letter "m" ($(u, v)^m$), all the other edges are called solid edges. The meta-edges are used to indicate the cancellation of the hypothesis;*

3. *there are three kinds of links: simple, focussing and defocussing, divided into logical and structural ones;*
4. *every vertex in a proof-graph is labelled with a conclusion of a unique link and is the premise of at most one link.*

The set of edges in a proof-graph may be empty. In this case the proof-graph represents an axiom.

In Fig. 2, we present the logical and structural links of N-Graphs.

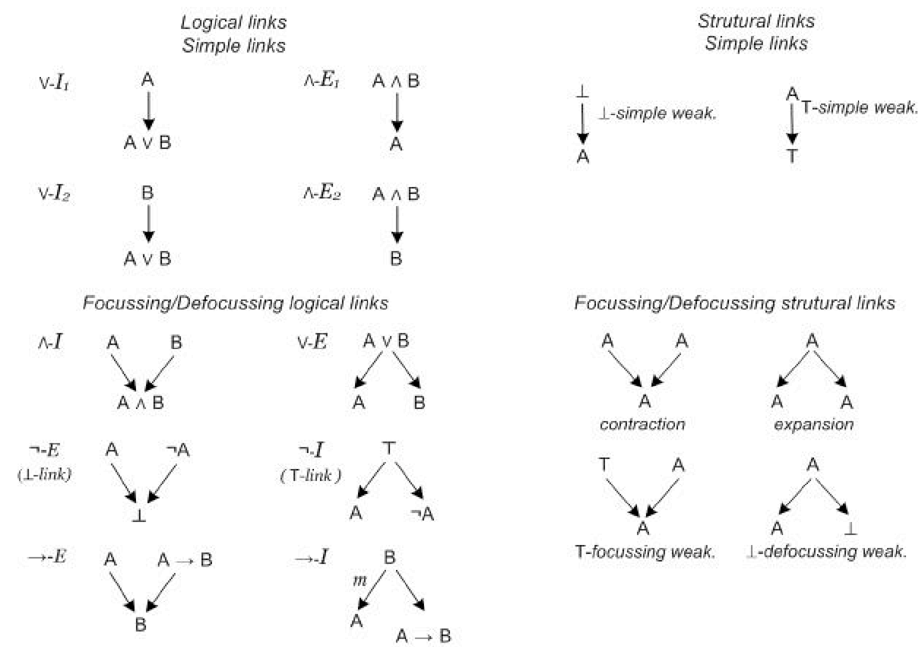


Fig. 2. N-Graphs links

We will also need the following definitions:

Definition 4 (Solid indegree/outdegree of v in a proof-graph). *The solid* indegree *of a vertex v in a proof-graph is the number of solid edges oriented towards it. The solid* outdegree *of a vertex v in a proof-graph is the number of solid edges oriented away from it.*

Notice that expansion and contraction links (under Focussing/Defocussing structural links in Figure 2) are also called switching links. And their edges are respectively called switching edges. The set of vertices with outdegree equal to zero is the set of conclusions of a derivation represented in a proof-graph G and is written as $\mathsf{CONC(G)}$. And the set of vertices with indegree equal to zero is the set of premisses of G, and it is written as $\mathsf{PREM(G)}$. The set of hypotheses of the graph $\mathsf{HYPO(G)}$ is the set of vertices in G with solid indegree equal to zero, but meta indegree equal to 1, denoting the set of cancelled hypothesis.

We can construct proof-graphs mirroring a sequent derivation. Basically for each inference rule in a sequent proof, we use the corresponding logical and structural links given in Fig. 2. The meta-edges are used to indicate the cancellation of hypotheses, when necessary. This process can be carried out for any LK derivation is the contents of the following theorem.

Theorem 1 (Map to N-Graphs). *Given a derivation π of $A_1, ..., A_n \vdash B_1, ..., B_m$ of the* LK *sequent calculus, it is possible to build a corresponding N-Graph $NG(\pi)$ whose elements of* $\mathsf{PREM}(NG(\pi))$ *and* $\mathsf{CONC}(NG(\pi))$ *are in one-to-one correspondence with the ocurrences of formulae $A_1, ..., A_n$ and $B_1, ..., B_m$ respectively.*

On the other hand, from N-graphs we can construct sequent derivations. This result is given by the following theorem.

Theorem 2 (Sequentialization). *Given an N-Graph derivation G, there is a derivation in the* LK *sequent calculus SC(G) of $A_1, ..., A_n \vdash B_1, ..., B_m$, whose ocurrences of formulae $A_1, ..., A_n$ and $B_1, ..., B_m$ are in one-to-one correspondence with the elements of sets* $\mathsf{PREM}(G)$ *and* $\mathsf{CONC}(G)$ *respectively.*

2.2 Soundness criterion

In order to determine whether an N-Graph corresponds to a proof-graph logically correct, one must remove an edge from every link of *expansion* and *contraction* (see Fig. 2) in this graph. Thus, new proof-graphs are generated, which must be acyclic and connected to be a derivation logically valid. Below we recall the geometric criteria formally established by Oliveira [dO01] to test whether a proof-graph is logically correct:

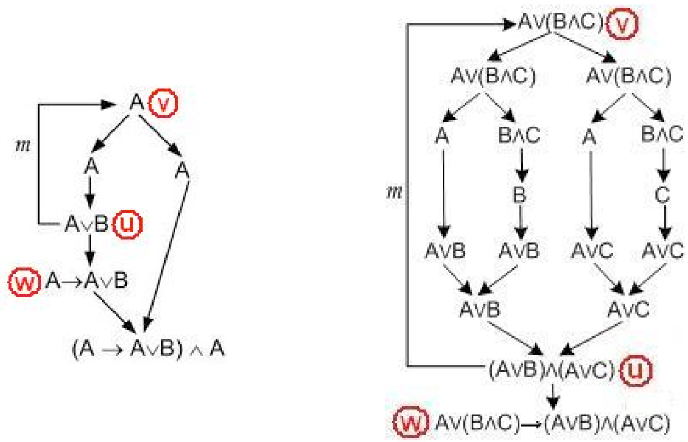
Definition 5 (Switching). *Given a proof-graph G, a switching graph S associated with G is a spanning subgraph[4] of G in which the following edges are removed:*

- *one of the two edges of every* expansion *and* contraction *link;*
- *all meta-edges.*

Definition 6 (Switching expansion). *Given a proof-graph G, a switching expansion graph S' associated with G is a spanning subgraph of G in which all meta-edges are removed as well as one of the two edges of every expansion link is removed.*

Definition 7 (Meta-condition). *Given a proof-graph G, we say that the meta-condition holds for it iff for every meta-edge $(u, v)^m$ of a defocussing link $\{(u, v)^m, (u, w)\}$ in G, there is a path or a semipath from v to u in every switching expansion graph S' associated with G, and that path can not pass through (u, w). Moreover, the solid indegree of v is equal to* zero*.*

Table 1. Examples of N-Graphs



Definition 8 (N-Graphs derivation). *A proof-graph G is an N-Graph derivation, iff the meta-condition holds for G and every switching graph associated with G is acyclic and connected.*

Looking at the examples shown in Table 1, we realize that in the graph at left of this figure we have the formula $A \vee B$ labelling the vertex u, A labelling the vertex v and $A \rightarrow A \vee B$ the vertex w. Thus, applying the meta-condition, we notice that v has solid indegree equal to *zero*. However, there is a switching graph expansion associated with this graph where there is no path or semipath from v to u without passing via (u, w). This happens because the only potential path is formed by an expansion link, as we know we shall remove one of the two edges of every expansion link. As a result, this graph does not satisfy the meta-condition. Now looking the graph at right of the same table, we check that in the above meta-edge we have the formula $(A \vee B) \wedge (A \vee C)$ as u, $A \vee (B \wedge C)$ as v and $A \vee (B \wedge C) \rightarrow ((A \vee B) \wedge (A \vee C))$ as w. Applying the meta-condition, we determine that v has solid indegree equal to *zero* and obtain a straight path from v to u. As a consequence, this graph does satisfy the meta-conditon and it is indeed an N-Graph derivation.

3 Intuitionistic N-Graphs

We now propose a version of N-Graphs for intuitionistic logic based on the sequent calculus *LJ'* of Maehara and prove soundness and completeness for this system.

We use the same propositional language of N-Graphs, that is, the logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\neg$ (negation), and the constants $\perp$ for *absurdity* (or *falsehood*) and $\top$ for *truth*.

[4] A spanning subgraph is a subgraph $G1$ of G containing all the vertices of G.

We also use the set of links of the N-Graphs (see Fig. 2) that represent the atomic steps in a derivation, but clearly we need to modify them somewhat to ensure compliance with the intuitionistic features of the system.

Intuitionistic logic is a sublogic of classical logic, as in classical logic we can draw inferences that we cannot in intuitionistic logic. Specifically, intuitionistic logic rejects the principle of excluded middle ($A \vee \neg A$). This principle is clearly represented in N-Graphs by the logical $\top$-*link* (see Fig. 2), so we remove it and introduce a new link $\neg$-I. More importantly, we establish a **restriction** in the links $\neg$-I and $\rightarrow$-I described below.

First recall that just like in N-Graphs, the construction of an intuitionistic N-Graphs derivation follows the definition of a proof-graph (see Definition 3) and must be constructed from the set of logical and structural links illustrated in Fig. 3. Note the restriction on the links $\neg$-I and $\rightarrow$-I. When applying those links, u must be the only conclusion at the time of its addition to the intuitionistic N-Graph G under construction. This restriction is similar to that of the calculus *LJ'*, which restricts the rules $R\neg$ and $R \rightarrow$.

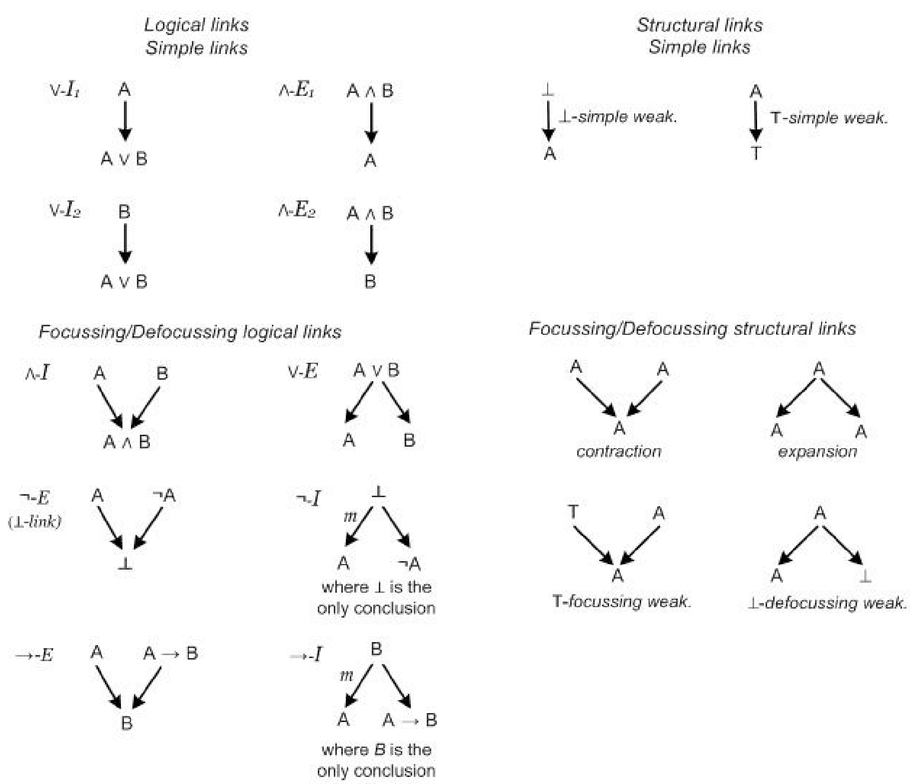


Fig. 3. Intuitionistic N-Graphs based on *LJ'*

Proof-graphs, like proof structures in the case of proof-nets, can be unsound. To determine the logically correct proof-graphs, the true intuitionistic N-Graphs, we need a global soundness criterion (see [dO01] and [Alv09]). These problems come from the multiple-conclusion structure of the graphs, which imply derivations with cycles that can be not logically correct. Thus we need a global soundness criterion to determine whether the proof-graph is, in fact, an intuitionistic N-Graph derivation or not.

3.1 Soundness criterion

The global soundness criterion for intuitionistic N-Graphs is obtained by changing two definitions in Section 2.2. The other definitions will remain the same.

Definition 9 (Intuitionistic meta-condition). *Given a proof-graph G, we say that the intuitionistic meta-condition holds for it iff for every meta-edge $(u,v)^m$ of a defocussing link $\{(u,v)^m,(u,w)\}$ in G:*

i) *there is a path or a semipath from v to u in every switching expansion graph S' associated with G, and that path can not pass through (u,w). Moreover, the solid indegree of v is equal to* zero*;*
ii) *and every path that starts from any vertex in* PREM*(G) goes to any vertex in* CONC*(G).*

Definition 10 (Intuitionistic N-Graph derivation). *A proof-graph G is an intuitionistic N-Graph derivation, iff the intuitionistic meta-condition holds for G and every switching graph associated with G is acyclic and connected.*

Two examples of an intuitionistic N-Graph derivation side-by-side with a corresponding sequent derivation are shown in Table 2 and Table 3. Applying the meta-condition, we see that these graphs do satisfy the conditon and they are intuitionistic N-Graphs.

Table 2. Intuitionistic N-Graph of $\neg\neg\neg A \to \neg A$

Derivation in *LJ'*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A}{A, \neg A \vdash}\,L\neg}{A \vdash \neg\neg A}\,R\neg}{A, \neg\neg\neg A \vdash}\,L\neg}{\neg\neg\neg A \vdash \neg A}\,R\neg}{\vdash \neg\neg\neg A \to \neg A}\,R\to$$

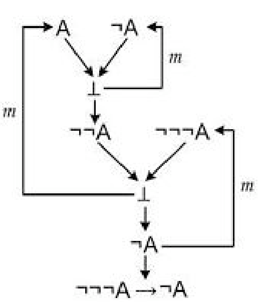
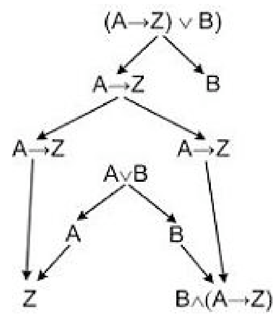


Table 3. Intuitionistic N-Graphs of $A \vee B, (A \to Z) \vee B \vdash Z, B \wedge (A \to Z), B$

Derivation in *LJ'*

$$\dfrac{B \vdash B \qquad \dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad Z \vdash Z}{A, A \to Z \vdash Z}\,L\to \qquad B \vdash B}{A \to Z, A \vee B \vdash B, Z}\,L\vee \qquad A \to Z \vdash A \to Z}{A \vee B, A \to Z, A \to Z \vdash Z, B \wedge (A \to Z)}\,R\wedge}{A \vee B, A \to Z \vdash Z, B \wedge (A \to Z)}\,LC}{A \vee B, (A \to Z) \vee B \vdash Z, B \wedge (A \to Z), B}\,L\vee$$



4 Soundness and completeness for Intuitionistic N-Graphs

In order to ensure that every intuitionistic N-Graph derivation represents a logically correct proof we need to prove soundness and completeness of the system. Soundness is proved through Theorem 3, while completeness is established through Theorem 2.

Theorem 3 (Map to intuitionistic N-Graphs). *Given a derivation π of $A_1, ..., A_n \vdash B_1, ..., B_m$ in the* LJ' *calculus, it is possible to build a corresponding intuitionistic N-Graph $NG(\pi)$ whose elements of* $\mathsf{PREM}(NG(\pi))$ *and* $\mathsf{CONC}(NG(\pi))$ *are in one-to-one correspondence with the ocurrences of formulae $A_1, ..., A_n$ and $B_1, ..., B_m$ respectively.*

Theorem 4 (Sequentialization). *Given an intuitionistic N-Graph derivation G, there is a derivation in the intuitionistic sequent calculus* LJ' *SC(G) of $A_1, ..., A_n \vdash B_1, ..., B_m$, whose ocurrences of formulae $A_1, ..., A_n$ and $B_1, ..., B_m$ are in one-to-one correspondence with the elements of sets* $\mathsf{PREM}(G)$ *and* $\mathsf{CONC}(G)$ *respectively.*

To prove these theorems we follow the procedure given by De Oliveira in [dO01]. The proofs of these theorems are presented in Appendix A.

5 Conclusions

We extended N-Graphs that were proposed for classical propositional logic, to a system that caters for *intuitionistic* propositional logic. N-Graphs are a multiple-conclusion system, so we rely on a basic intuitionistic sequent calculus with multiple-conclusions. In this paper we used Mahera's system LJ' as our basic multiple conclusion intuitionstic sequent calculus, as it seemed the closest to the original N-Graphs.

We have proved soundness and completeness for this system, which we call *iN-Graphs*, for intuitionistic N-Graphs, by a modification of the original proof for classical N-Graphs. In future work we intend to describe and prove sound and complete intuitionistic N-Graphs based on the system *FIL* of Full Intuitionistic Logic, introduced by de Paiva and Pereira, [PP05] mentioned in the introduction. While the system *LJ'* is a sound basis for our constructive geometric explorations, its ability to deal with cut-free proofs is somewhat limited, as its process of cut-elimination throws derivations out of the system([Sch91]). We hope to obtain better behavior of the process of cut-elimination (and of its geometric features) using the system FIL. FIL and LJ' need to look always the same, not math mode sometimes and other times not. We have a draft of an intuitionistic system for N-Graphs with a indexing device that allows us to record the premisses on which each formula in the graph depends on, and thus to correctly apply the links $\neg$-*I* and $\rightarrow$-*I* and determine the soundness of the intuitionistic N-Graphs. We construct the proof of completeness of the system by mapping the intuitionistic N-Graphs to *FIL*, while for the proof of soundness we hope it can be done by sequentialization over *FIL*. But this remains future work. Also future work is to show how one can normalize *iN-Graphs* and how this normalization relates to cut-elimination in the sequent calculus.

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A Proof of Theorems 2 and 3

A.1 Proof of Theorem 2

Proof. We prove completeness by induction on the structure of the derivation π but in the calculus *LJ'*, whose equivalence with LJ was proved by Maehara [Mae54]. We follow the procedure given by Oliveira in [dO01], and due to the similarity of the proofs, of the 13 cases we present only the different cases for this theorem. Without loss of generality, when the antecedent (succedent) of $\vdash$ is empty we use the constant $\top$ ($\perp$).

- **Case 3:** If Π is obtained from Π_1 by $R\neg$:

$$\dfrac{\begin{array}{c}\Pi_1\\ A_2, ..., A_n, \mathbf{A} \vdash\end{array}}{A_2, ..., A_n \vdash \neg\mathbf{A}}\ R\neg$$

$NG(\Pi)$ is like in Fig. 5.

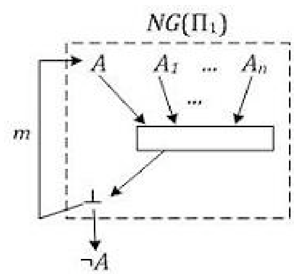

Fig. 4. $NG(\Pi)$ case 3

By induction hypothesis we have obtained $NG(\Pi_1)$. $NG(\Pi)$ is obtained by adding the vertices labeled $\perp$ and $\neg A$ as well as the meta-edge $(\perp, A)$ and the edge $(\perp, \neg A)$ to $NG(\Pi_1)$.
For $NG(\Pi)$ be an iN-Graph, the intuitionistic meta-condition should hold. It is obvious that the solid indegree of A is equal to zero and $\perp$ is the only conclusion in $NG(\Pi_1)$, however the other conditions need to be analyzed carefully:

1. A and $\perp$ are linked in $NG(\Pi_1)$ (through a path or semi-path). Otherwise, $NG(\Pi_1)$ would be disconnected;
2. If A is a branch point of an *expansion* link, then both conclusions of this link are also connected to $\perp$ (through a path or semi-path). Otherwise, a switching graph associated with $NG(\Pi_1)$ would be disconnected;
3. Then we conclude that in every switching expansion graph associated with $NG(\Pi)$ there is a path or semi-path from A to $\perp$.

– **Case 8:** If Π is as follows:

$$\dfrac{\begin{array}{c}\Pi_1 \\ A_1, ..., A_n, \mathbf{A} \vdash \mathbf{B}\end{array}}{A_1, ..., A_n \vdash \mathbf{A} \rightarrow \mathbf{B}} R\rightarrow$$

$NG(\Pi)$ is like in Fig. 10.

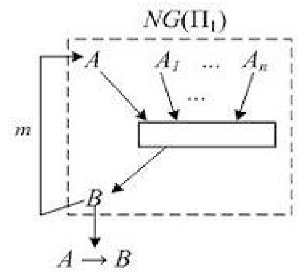

Fig. 5. $NG(\Pi)$ case 8

For $NG(\Pi)$ be an iN-Graph, the intuitionistic meta-condition should hold. It is obvious that the solid indegree of A is equal to zero and that B is the only conclusion in $NG(\Pi_1)$, however the other conditions need to be considered carefully:

1. A and B are linked in $NG(\Pi_1)$ (through a path or semi-path). Otherwise, $NG(\Pi_1)$ would be disconnected;
2. if A is a branch point of one *expansion* link, then both conclusions of this link are also linked to B (through a path or semi-path). Otherwise, a switching expansion graph with $NG(\Pi_1)$ would be disconnected.
3. Then we conclude that in every switching expansion graph associated with $NG(\Pi)$ there is a path or semi-path from A to B.

A.2 Proof of Theorem 3

Proof. It is similar to the proof for N-graphs. We need the following additional definitions.

Definition 11 (Conjunctive/Disjunctive link). *The links $\wedge$-I, $\neg$-E, $\rightarrow$-E, $\top$-$focussing$ weak. and expansion link are called conjunctive. The links $\vee$-E, $\neg$-I, $\rightarrow$-I, $\bot$-$defocussing$ weak. and contraction link are called disjunctive.*

Definition 12 (Initial (Final) simple link). *A simple link (u, v) in a given iN-Graph G is initial (final) if $u \in \mathsf{PREM}(G)$ ($v \in \mathsf{CONC}(G)$).*

Definition 13 (Initial (Final) defocussing link). *A defocussing link $\{(u, v_1), (u, v_2)\}$ with the exception of $\rightarrow$-I and $\neg$-I, in a given iN-Graph G is initial (final) if $u \in \mathsf{PREM}(G)$ ($\{v_1, v_2\} \subset \mathsf{CONC}(G)$).*

Definition 14 (Initial (Final) focussing link). *A focussing link* $\{(u_1, v), (u_2, v)\}$ *is initial (final) in a given iN-Graph* G*, if* $\{u_1, u_2\} \in \mathsf{PREM}(G)(v \in \mathsf{CONC}(G))$

Definition 15 (Final $\rightarrow$-I ($\neg$-I)). *A link* $\rightarrow$*-*I *(*$\neg$*-*I*)* $\{(u, v_1)^m, (u, v_2)\}$ *in a given iN-Graph* G *is final if* $v_2 \in \mathsf{CONC}(G)$.

Definition 16 (Partition property). *Let* G' *an spanning subgraph of* G *obtained through elimination of a focussing link* $\{(u_1, v), (u_2, v)\}$ *or through the elimination of a defocussing link* $\{(u, v_1), (u, v_2)\}$*. If* G' *has three disjoint components, we say that the focussing or defocussing link removed has the partition property.*

Definition 17 (Cut point of branching). *We say that a vertex* v *is a cut point of branching if it is both a branch point of an* contraction *or* expansion *link.*

The proof of the Theorem 3 proceeds by induction on the number of vertices of a given iN-Graph G. We provide an algorithm to transform an iN-Graph G in a derivation Π of the corresponding calculus LJ'.

Without loss of generality, we assume $\perp$ like $A \wedge \neg A$, where the formula A belongs to the premise or conclusion of the link;

The proof is as follows:

1. If G has only one vertex v labeled with A. It is immediate: $CS(G)$ is $A \vdash A$;
2. If there exists a simple link (initial or final) (u, v) in G, then we should remove it as follows :
   a) If (u, v) is initial and is one of the links $\wedge$-E_1, $\wedge$-E_2 or $\perp$-*simple weak.*, then when we remove from G, two subgraphs which are clearly iN-Graphs are obtained:
      i) the vertex u;
      ii) the iN-Graph G_1 with $v \in \mathsf{PREM}(G_1)$. In the case of $\wedge$-E_1, G_1 is like the iN-Graph in the dashed box shown in Fig. 6.

   With the induction hypothesis is built $CS(G_1)$ as follows in the case of $\wedge$-E_1:

$$\begin{array}{c} \Pi_1 \\ A_2, ..., A_n, \mathbf{A} \vdash B_1, ..., B_m \end{array}$$

   Therefore, $CS(G)$ is built by $L\wedge_1$ applied to $CS(G_1)$:

$$\dfrac{\begin{array}{c} \Pi_1 \\ A_2, ..., A_n, \mathbf{A} \vdash B_1, ..., B_m \end{array}}{A_2, ..., A_n, \mathbf{A} \wedge \mathbf{B} \vdash B_1, ..., B_m}\, L\wedge_1$$

   Similarly, we build $CS(G)$ from $CS(G_1)$ in the case of $\wedge$-E_2 and $\perp$-*simple weak.* through $L\wedge_2$ and RW, respectively.

   b) If (u, v) is final and is one of the links $\vee$-I_1, $\vee$-I_2 or $\top$-*simple weak.*, then when removed from G, two subgraphs which are clearly iN-graphs are obtained:
      i) the vertex v;

ii) The iN-Graph G_1 with $u \in \mathsf{CONC}(G_1)$. In the case of $\vee$-I_1, G_1 is like the iN-Graph dashed box shown in Fig. 8.

With the induction hypothesis is built $CS(G_1)$ as follows in the case of $\vee$-I_1:

$$\begin{array}{c} \Pi_1 \\ A_1, ..., A_n \vdash \mathbf{A}, B_2, ..., B_m \end{array}$$

Therefore, $CS(G)$ is built by applying $R\vee_1$ to $CS(G_1)$:

$$\dfrac{\begin{array}{c} \Pi_1 \\ A_1, ..., A_n \vdash \mathbf{A}, B_2, ..., B_m \end{array}}{A_1, ..., A_n \vdash \mathbf{A} \vee \mathbf{B}, B_2, ..., B_m}\, R\vee_1$$

The other cases are similar.

c) The cases where (u, v) is final and (i) is one of the links $\wedge$-E_1, $\wedge$-E_2 or $\perp$-*simple weak.*; or (ii) is initial and is one of the links $\vee$-I_1, $\vee$-I_2 or $\top$-*simple weak.*, we need to use the cut rule to build $CS(G)$. For example, if (u, v) is the final link $\wedge$-E_2, then by removal of G, the resulting subgraphs v and G_1 (with $u \in \mathsf{CONC}(G_1)$) are clearly iN-Graphs. The induction hypothesis build $CS(G_1)$ as follows:

$$\begin{array}{c} \Pi_1 \\ A_1, ..., A_n \vdash \mathbf{A} \wedge \mathbf{B}, B_2, ..., B_m \end{array}$$

$CS(G)$ is built as follows:

$$\dfrac{\begin{array}{c} \Pi_1 \\ A_1, ..., A_n \vdash \mathbf{A} \wedge \mathbf{B}, B_2, ..., B_m \end{array} \qquad \dfrac{B \vdash B}{\mathbf{A} \wedge \mathbf{B} \vdash B}\, L\wedge_2}{A_1, ..., A_n \vdash B, B_2, ..., B_m}\, corte$$

The other cases are similar.

3. If $\{(u, v_1)^m, (u, v_2)\}$ is final and is one of the links $\rightarrow$-I or $\neg$-I. Then, when we remove it from G, i.e. the edges $(u, v_1)^m$ and (u, v_2) as well as the vertex v_2, two subgraphs are obtained, otherwise there would be one path or semi-path from v_1 to u, they are:
   i) the vertex v_2;
   ii) the iN-Graph G_1 with $v_1 \in \mathsf{PREM}(G_1)$ and $u \in \mathsf{CONC}(G_1)$. G_1 is clearly an iN-Graph. It is illustrated in the dashed box of Fig. 10 for the case $\rightarrow$-I and in the Fig. 5 for the case $\neg$-I.

   The induction hypothesis build $CS(G_1)$ as follows in the case of $\rightarrow$-I:

$$\begin{array}{c} \Pi_1 \\ A_1, ..., A_n, \mathbf{A} \vdash \mathbf{B} \end{array}$$

   Therefore, $CS(G)$ is deducted from $CS(G_1)$ by $R \rightarrow$:

$$\dfrac{\begin{array}{c} \Pi_1 \\ A_1, ..., A_n, \mathbf{A} \vdash \mathbf{B} \end{array}}{A_1, ..., A_n \vdash \mathbf{A} \rightarrow \mathbf{B}}\, R \rightarrow$$

   The other case is as follows:

$$\begin{array}{c} \Pi_1 \\ A_1, ..., A_n, \mathbf{A} \vdash \end{array}$$

Therefore, $CS(G)$ is deducted from $CS(G_1)$ by $R\neg$:

$$\dfrac{\begin{array}{c} \Pi_1 \\ A_1, ..., A_n, \mathbf{A} \vdash \end{array}}{A_1, ..., A_n \vdash \neg\mathbf{A}}\ R\neg$$

4. If there is any initial link $\{(u_1, v), (u_2, v)\}$ in G, with the exception of *contraction*, where u_1 and u_2 belong to $\mathsf{PREM}(G)$:
   a) If $\{(u_1, v), (u_2, v)\}$ is $\rightarrow$-E then after removing it from G, three subgraphs that are clearly iN-Graphs are obtained:
   i) the vertices u_1 and u_2 (two iN-Graphs);
   ii) and the iN-Graph G_1 with $v \in \mathsf{PREM}(G_1)$.
   Let A_1 and A_2 in G occurrences of the formula A and $A \rightarrow B$, respectively. The induction hypothesis build $CS(G_1)$ as follows:

$$\begin{array}{c} \Pi_1 \\ A_3, ..., A_n, \mathbf{B} \vdash B_1, ..., B_m \end{array}$$

   So, for $CS(G)$ we can select the derivation:

$$\dfrac{A \vdash \mathbf{A} \qquad \begin{array}{c} \Pi_1 \\ A_3, ..., A_n, \mathbf{B} \vdash B_1, ..., B_m \end{array}}{A, A_3, ..., A_n, \mathbf{A} \rightarrow \mathbf{B} \vdash B_1, ..., B_m}\ L\rightarrow$$

   b) If $\{(u_1, v), (u_2, v)\}$ is one of the focussing links $\wedge$-I, $\bot$-*link* or $\top$-*focussing weak.*, $CS(G)$ is built using the cut rule.
   b.1) The cases where the link is $\wedge$-I or $\bot$-*link* are similar. For example, we use the link $\wedge$-I. After removing from G, the resulting subgraphs u_1, u_2 and G_1 (with $v \in \mathsf{PREM}(G_1)$) are clearly iN-Graphs. Let A_1 and A_2 occurrences of formulas A and B respectively, the induction hypothesis has built $CS(G_1)$ as follows:

$$\begin{array}{c} \Pi_1 \\ A_3, ..., A_n, \mathbf{A} \wedge \mathbf{B} \vdash B_1, ..., B_m \end{array}$$

   So, for $CS(G)$ we can select the derivation:

$$\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\ R\wedge \qquad \begin{array}{c} \Pi_1 \\ A_3, ..., A_n, \mathbf{A} \wedge \mathbf{B} \vdash B_1, ..., B_m \end{array}}{A, B, A_3, ..., A_n \vdash B_1, ..., B_m}\ cut$$

   b.2) the cases where the link is $\top$-*focussing weak.*, for $CS(G)$ we can select the derivation:

$$\dfrac{\dfrac{A \vdash A \qquad \dfrac{A \vdash A}{\neg A, A \vdash}\ L\neg}{A \vee \neg A, A \vdash A}\ L\vee \qquad \mathbf{A}, A_3, ..., A_n \vdash B_1, ..., B_m}{A \vee \neg A, A, A_3, ..., A_n \vdash B_1, ..., B_m}\ cut$$

5. If there is any final defocussing link $\{(u, v_1), (u, v_2)\}$ (with the exception of the *expansion* and $\rightarrow$-I links) in G, where v_1 and v_2 belong to $\mathsf{CONC}(G)$, we obtain three iN-Graphs after removing it: the vertex v_1; the vertex v_2; and the iN-Graph G_1 with $u \in \mathsf{CONC}(G_1)$. Here we need to use the *cut* rule. For example, the case where the final link is $\vee$-E, $CS(G_1)$ is as follows:

$$\begin{array}{c} \Pi_1 \\ A_1, ..., A_n \vdash \mathbf{A} \vee \mathbf{B}, B_3, ..., B_m \end{array}$$

So for $CS(G)$ we can select the derivation:

$$\dfrac{\begin{array}{c}\Pi_1 \\ A_1, ..., A_n \vdash \mathbf{A} \vee \mathbf{B}, B_3, ..., B_m\end{array} \qquad \dfrac{A \vdash A \qquad B \vdash B}{\mathbf{A} \vee \mathbf{B} \vdash A, B}\,L\vee}{A_1, ..., A_n \vdash A, B, B_3, ..., B_m}\,cut$$

6. If there is any final *contraction* link $\{(u_1, v), (v, u_2)\}$ in G where $v \in \mathsf{CONC}(G)$, then after removing it from G, only two subgraphs are obtained. Otherwise, G would not be an iN-graph because an switching subgraph associated with it would be disconnected. The subgraphs are:
i) the vertex v;
ii) the iN-Graph G_1 with u_1 and u_2 in $\mathsf{CONC}(G_1)$
$CS(G_1)$ is as follows:

$$\begin{array}{c} \Pi_1 \\ A_1, ..., A_n \vdash \mathbf{A}, \mathbf{A}, B_2, ..., B_m \end{array}$$

So, $CS(G)$ is deducted from $CS(G_1)$ by RC:

$$\dfrac{\begin{array}{c}\Pi_1 \\ A_1, ..., A_n \vdash \mathbf{A}, \mathbf{A}, B_2, ..., B_m\end{array}}{A_1, ..., A_n \vdash \mathbf{A}, B_2, ..., B_m}\,RC$$

7. If there is any initial *expansion* link $\{(u, v_1), (u, v_2)\}$ in G where $u \in \mathsf{PREM}(G)$. Then after removing it from G, for the same reasons the previous case, two subgraphs are obtained:
i) the vertex u;
ii) and the iN-Graph G_1 with v_1 and v_2 in $\mathsf{PREM}(G_1)$
$CS(G_1)$ is as follows:

$$\begin{array}{c} \Pi_1 \\ A_2, ..., A_n, \mathbf{A}, \mathbf{A} \vdash B_1, ..., B_m \end{array}$$

So, $CS(G)$ is deducted from $CS(G_1)$ by LC:

$$\dfrac{\begin{array}{c}\Pi_1 \\ A_2, ..., A_n, \mathbf{A}, \mathbf{A} \vdash B_1, ..., B_m\end{array}}{A, A_2, ..., A_n \vdash B_1, ..., B_m}\,LC$$

8. If each initial link is focussing and disjunctive, and each final link is defocussing and conjunctive. This case further complicated by the presence of a large number of links and situations where we need to divide our iN-Graph into two disjoint iN-Graphs, is treated by De Oliveira [dO01], formulating and proving the partition theorem that shown below.

   Theorem 5 (Partition, De Oliveira). *Let G an iN-graph where each initial link is focussing and disjunctive and each final link is focussing and conjunctive. So there must be or some*
   (i) initial defocussing and conjunctive link or final focussing and conjunctive link with the partition property or (ii) one cut point of branching.

   Corollary 1. *If G is an iN-Graph as in Theorem 4 then:*
   (a) If $\{(u, v_1), (u, v_2)\}$ is an initial defocussing and disjunctive link with the partition property, then after removing it from G, we obtain three subgraphs G_1, G_2 and G_3, which are also iN-Graph as follows:
   If $\{(u, v_1), (u, v_2)\}$ is an initial focussing and disjunctive link with the property of partition, then remove it from the G we obtain three subgraphs G_1, G_2 and G_3 , which are also iN-Graph as follows:
   - *G_1 is the vertex u;*
   - *G_2 has v_1 between its premisses;*
   - *G_3 has v_2 between its premisses;*
   - $\mathsf{PREM}(G) = u \cup \mathsf{PREM}(G_2) \cup \mathsf{PREM}(G_3)$;
   - $\mathsf{CONC}(G) = \mathsf{CONC}(G_2) \cup \mathsf{CONC}(G_3)$.

   (b) If $\{(u_1, v), (u_2, v)\}$ is an final focussing and conjunctive link with the partition property, then remove it from G we obtain three subgraphs G_1, G_2 and G_3, which are also iN-Graph as follows:
   - *G_1 is the vertex v;*
   - *G_2 has u_1 between its conclusions;*
   - *G_3 has u_2 between its conclusions;*
   - $\mathsf{PREM}(G) = \mathsf{PREM}(G_2) \cup \mathsf{PREM}(G_3)$;
   - $\mathsf{CONC}(G) = u \cup \mathsf{CONC}(G_2) \cup \mathsf{CONC}(G_3)$.

   (c) If s is a cut point of branching of an expansion *link $\{(s, s_1), (s, s_2)\}$, then remove it from G we obtain two subgraphs which are also iN-Graph as follows:*
   - *G_1 has s between its conclusions;*
   - *G_2 has s_1 and s_2 between its conclusions;*
   - $\mathsf{PREM}(G) = \mathsf{PREM}(G_1) \cup \mathsf{PREM}(G_2)$;
   - $\mathsf{CONC}(G) = \mathsf{CONC}(G_1) \cup \mathsf{CONC}(G_2)$.

   Proof. If G is an iN-Graph, it is immediate to verify that the subgraphs as constructed in the different cases, also follow the global soundness criterion.

Continuing the proof, since the Theorem 2 we have that in the Case 8 there is a defocussing/focussing link with the partition property or or there is a cut point of branching in G. The Corollary 1 said if a focussing/defocussing link has the partition property, then G can be divided into three disjoint proof-graphs, which are also iN-Graph. Then:

1. In the case $\{(u, v_1), (u, v_2)\}$ is the defocussing link with the partition property. G_1 is the vertex u, and G_2 and G_3 have v_1 and v_2 between its premisses, respectively. The induction hypothesis has built $CS(G_2)$ and $CS(G_3)$. Let the link $\vee$-E and $\{A \vee B\}$, Δ_1, Δ_2 three partitions of $A_1, ..., A_n$ as well as Γ_1, Γ_2 two partitions of $B_1, ..., B_m$, then $CS(G_2)$ and $CS(G_3)$ are as follows:

$$\begin{array}{c} \Pi_2 \\ \Delta_1, \mathbf{A} \vdash \Gamma_1 \end{array}$$

$$\begin{array}{c} \Pi_3 \\ \Delta_2, \mathbf{B} \vdash \Gamma_2 \end{array}$$

So, $CS(G)$ is built from $CS(G_2)$ and $CS(G_3)$ by $L\vee$:

$$\dfrac{\begin{array}{c} \Pi_2 \\ \Delta_1, \mathbf{A} \vdash \Gamma_1 \end{array} \qquad \begin{array}{c} \Pi_3 \\ \Delta_2, \mathbf{B} \vdash \Gamma_2 \end{array}}{\Delta_1, \Delta_2, \mathbf{A} \vee \mathbf{B} \vdash \Gamma_1, \Gamma_2} \, L\vee$$

2. In the case $\{(u_1, v), (u_2, v)\}$ is the focussing link with the partition property. G_1 is the vertex v, and G_2 and G_3 has u_1 and u_2 between its conclusions, respectively. The induction hypothesis build $CS(G_2)$ and $CS(G_3)$. Let the link $\wedge$-I and $\{A \wedge B\}$, Γ_1, Γ_2 three partitions of $B_1, ..., B_n$ as well as Δ_1, Δ_2 two partitions of $A_1, ..., A_m$, then $CS(G_2)$ and $CS(G_3)$ are as follows:

$$\begin{array}{c} \Pi_2 \\ \Delta_1 \vdash \mathbf{A}, \Gamma_1 \end{array}$$

$$\begin{array}{c} \Pi_3 \\ \Delta_2 \vdash \mathbf{B}, \Gamma_2 \end{array}$$

So, $CS(G)$ is built from $CS(G_2)$ and $CS(G_3)$ by $R\wedge$:

$$\dfrac{\begin{array}{c} \Pi_2 \\ \Delta_1 \vdash \mathbf{A}, \Gamma_1 \end{array} \qquad \begin{array}{c} \Pi_3 \\ \Delta_2 \vdash \mathbf{B}, \Gamma_2 \end{array}}{\Delta_1, \Delta_2 \vdash \mathbf{A} \wedge \mathbf{B}, \Gamma_1, \Gamma_2} \, R\wedge$$

3. In the case s is a cut point of branching of the *expansion* link $\{(s, s_1), (s, s_2)\}$. Let s labeled with A and Δ_1, Δ_2 and Γ_1, Γ_2 partitions of $A_1, ..., A_n$ and $B_1, ..., B_m$, respectively. The induction hypothesis build $CS(G_1)$ and $SC(G_2)$ respectively as follows:

$$\begin{array}{c} \Pi_1 \\ \Delta_1 \vdash \mathbf{A}, \Gamma_1 \end{array}$$

$$\begin{array}{c} \Pi_2 \\ \Delta_2, \mathbf{A}, \mathbf{A} \vdash \Gamma_2 \end{array}$$

So, $CS(G)$ is built from $CS(G_2)$ and $CS(G_3)$ by *cuts* and *contrations* as follows:

$$
\dfrac{\dfrac{\dfrac{\Pi_1}{\Delta_1 \vdash \mathbf{A}, \Gamma_1} \quad \dfrac{\dfrac{\Pi_1}{\Delta_1 \vdash \mathbf{A}, \Gamma_1} \quad \dfrac{\Pi_2}{\Delta_2, \mathbf{A}, \mathbf{A} \vdash, \Gamma_2}}{\Delta_1, \Delta_2, \mathbf{A} \vdash \Gamma_1, \Gamma_2}\, cut}{\Delta_1, \Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_1, \Gamma_2}\, cut}{\Delta_1, \Delta_2 \vdash \Gamma_1, \Gamma_2}\, contractions
$$