Contextual Constructive Description Logics

Valeria de Paiva   Natasha Alechina

January 31, 2011

Abstract

Constructive modal logics come in several different flavours and constructive *description* logics, while much more recent and less studied, not surprisingly, do the same. After all, it is a well-known result of Schild that description logics are simply variants of n-ary basic modal logic. There are several extensions of classical description logics, with modalities, temporal assertions, etc. As far as we know there are no such extensions for constructive description logics. Hence this note is a formal description of the extension of a constructive description logic c$\mathcal{ALC}$ [MS08] with contexts as modalities, as described in [deP03a], following the blueprint of [WZ99].

1 Introduction

Description Logics are a knowledge representation formalism, much used in Artificial Intelligence. They are logic-based formalisms intended for representing knowledge about concept hierarchies, supplied with effective reasoning procedures and a declarative semantics. Description logics are very popular nowadays, perhaps due to their proposed applications in the Semantic Web. Most uses of description logics consider classical systems. However, considering versions of *constructive* description logics makes sense, both from a theoretical and from a practical viewpoint, as discussed in [deP03].

Description logics tend to be bundled in families of logical systems, depending on which concept *constructors* you allow in the logic. Since description logics came into existence as fragments of first-order logic, chosen to find the best trade-off possible between expressiveness and tractability of the fragment, several systems were discussed and eventually a taxonomy of systems emerged. In this taxonomy, the system called ALC (for Attributive Language with Complements) has come to be known as the canonical basic one. As far as *constructive* description logics are concerned, Mendler and Scheele have worked out a very compelling system, which they call $c\mathcal{ALC}$ ([MS08], based on the constructive modal logic

CK[BPR01]). A different constructive version of ALC, based on the framework for constructive modal logics developed by Simpson in his phd thesis [Sim95] was developed by Hausler, Rademaker and de Paiva. Their system, called iALC for Intuitionistic ALC, was described in [HRP10] and it is the reduced version of Braüner and de Paiva's system of Intuitionistic Hybrid logic, IHL [BdP06]. The systems $c\mathcal{ALC}$ and iALC are alternative formalizations of constructive description logics, and the main difference between these systems is whether they satisfy (or not) distribution of possibility over disjunction.

In this note, we start be recalling the description logic $c\mathcal{ALC}$. We then consider one extra modality on top of $c\mathcal{ALC}$, following the blueprint of [WZ99]. We discuss how this system $c\mathcal{ALC}_{\Box}$ could be extended to a framework of contexts as modalities, as described in [deP03a], to obtain the system $c\mathcal{ALC}_{ctx}$. We prove some basic properties of the unary system and discuss its intended application.

2 Constructive description logic $c\mathcal{ALC}$

The basic building blocks of description logics are *concepts*, *roles* and *individuals*. We think of concepts as unary predicates in usual first-order logic and of roles as binary predicates, used to modify the concepts. Like classical ALC [DL03] the intuitionistic version c$\mathcal{ALC}$ is a basic description language whose concept constructors are described by the following grammar:

$$C, D ::= A \mid \bot \mid \top \mid \neg C \mid C \sqcap D \mid C \sqcup D \mid C \sqsubseteq D \mid \exists R.C \mid \forall R.C$$

where A stands for an atomic concept and R for an atomic role. This syntax is more general than ALC as it includes subsumption $\sqsubseteq$ as a concept-forming operator.

Following Mendler and Scheele we say a constructive interpretation of c$\mathcal{ALC}$ is a structure $\mathcal{I} = (\Delta^{\mathcal{I}}, \preceq^{\mathcal{I}}, \bot^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of a non-empty set $\Delta^{\mathcal{I}}$ of entities in which each entity represents a partially defined individual; a refinement preordering $\preceq^{\mathcal{I}}$ on $\Delta^{\mathcal{I}}$, i.e., a reflexive and transitive relation; $\bot^{\mathcal{I}}$ is a subset of fallible entities satisfying $\bot$ (fallible entities are over-defined and hence self-contradictory. This set is closed under refinement, that is, $x \in \bot^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in \bot^{\mathcal{I}}$); and an interpretation function $\cdot^{\mathcal{I}}$ mapping each role name R to a binary relation $R^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ and each atomic concept A to a set $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$ which is closed under refinement, i.e., $x \in A^{\mathcal{I}}$ and $x \preceq^{\mathcal{I}} y$ implies $y \in A^{\mathcal{I}}$. The interpretation $\mathcal{I}$ is lifted from atomic $\bot, A$ to arbitrary concepts, where $\Delta^{\mathcal{I}}_c =_{df} \Delta^{\mathcal{I}} \setminus \bot^{\mathcal{I}}$ is the set of non-fallible elements, via:

$$
\begin{array}{rcl}
\top^{\mathcal{I}} & =_{df} & \Delta^{\mathcal{I}} \\
(\neg C)^{\mathcal{I}} & =_{df} & \{x|\forall y \in \Delta_c^{\mathcal{I}}.x \preceq y \Rightarrow y \notin C^{\mathcal{I}}\} \\
(C \sqcap D)^{\mathcal{I}} & =_{df} & C^{\mathcal{I}} \cap D^{\mathcal{I}} \\
(C \sqcup D)^{\mathcal{I}} & =_{df} & C^{\mathcal{I}} \cup D^{\mathcal{I}} \\
(C \sqsubseteq D)^{\mathcal{I}} & =_{df} & \{x|\forall y \in \Delta_c^{\mathcal{I}}.(x \preceq y \text{ and } y \in C^{\mathcal{I}}) \Rightarrow y \in D^{\mathcal{I}}\} \\
(\exists R.C)^{\mathcal{I}} & =_{df} & \{x|\forall y \in \Delta_c^{\mathcal{I}}.x \preceq y \Rightarrow \exists z \in \Delta^{\mathcal{I}}.(y,z) \in R^{\mathcal{I}} \text{ and } z \in C^{\mathcal{I}}\} \\
(\forall R.C)^{\mathcal{I}} & =_{df} & \{x|\forall y \in \Delta_c^{\mathcal{I}}.x \preceq y \Rightarrow \forall z \in \Delta^{\mathcal{I}}.(y,z) \in R^{\mathcal{I}} \Rightarrow z \in C^{\mathcal{I}}\}
\end{array}
$$

Semantic validity can be introduced as follows: say "x satisfies C in the interpretation $\mathcal{I}$", written as $\mathcal{I}, x \models C$, if x is in the interpretation of C, $x \in C^{\mathcal{I}}$. Say $\mathcal{I} \models C$ if this happens for all x in $\Delta^{\mathcal{I}}$. Finally say $\models C$ if for all interpretations $\mathcal{I}$ we have $\mathcal{I} \models C$. These definitions are usually extended to sets of concepts.

Typical reasoning in description logics is done via TBoxes and ABoxes. If we use Θ for a TBox, ie a collection of concepts and subsumptions and Γ for an ABox, a collection of instantiations of concepts then we can say $\Theta, \Gamma \models C$ if for all interpretations $\mathcal{I}$, which are models of all the concepts in Θ it is the case that every x in $\mathcal{I}$ which satisfy the axioms in Γ must also satisfy C, or

$$\forall \mathcal{I}.\forall x \in \Delta^{\mathcal{I}}.(\mathcal{I} \models \Theta \text{ and } \mathcal{I}, x \models \Gamma) \text{ implies } \mathcal{I}, x \models C$$

A Hilbert-style axiomatization of c$\mathcal{ALC}$ consists of all axioms and rules displayed in Figure 1.

(IPL) all axioms of propositional intuitionistic logic

$(\forall\mathsf{K})$ $\forall R.(C \sqsubseteq D) \sqsubseteq (\forall R.C \sqsubseteq \forall R.D)$

$(\exists\mathsf{K})\exists R.(C \sqsubseteq D) \sqsubseteq (\exists R.C \sqsubseteq \exists R.D)$

$(\mathsf{DISTmix})(\exists R.C \sqsubseteq \forall R.C) \sqsubseteq \forall R.(C \sqsubseteq D)$

(Nec) If C is a theorem then $\forall R.C$ is a theorem too.

(MP) If C and $C \sqsubseteq D$ are theorems, D is a theorem too.

Figure 1: The System c$\mathcal{ALC}$: Hilbert-style

Mendler and Scheele([MS08] p.7) proved:

Theorem 1 (Mendler-Scheele)**.** *The Hilbert calculus described in Figure 1 is sound and complete for TBox reasoning, that is* $\Theta, \emptyset \models C$ *if and only if* $\Theta \vdash_H C$.

A sequent calculus version of c$\mathcal{ALC}$ is given by Mendler and Scheele, who also prove cut-elimination for their calculus. But their calculus is a a tableau style calculus with positive and negative information about concepts and a simpler one is available, see Figure 2.

$$\frac{}{\Gamma, C \Rightarrow\ C, \Delta} \qquad \frac{}{\Gamma, \bot \Rightarrow\ C, \Delta}$$

$$\frac{\Gamma \Rightarrow\ C, \Delta \qquad \Gamma, D \Rightarrow\ \Delta}{\Gamma, C \sqsubseteq D \Rightarrow\ \Delta}\ \sqsubseteq\text{-l} \qquad \frac{\Gamma, C \Rightarrow\ D}{\Gamma \Rightarrow\ C \sqsubseteq D}\ \sqsubseteq\text{-r}$$

$$\frac{\Gamma, C, D \Rightarrow\ \Delta}{\Gamma, (C \sqcap D) \Rightarrow\ \Delta}\ \sqcap\text{-l} \qquad \frac{\Gamma \Rightarrow\ C, \Delta \qquad \Gamma \Rightarrow\ D, \Delta}{\Gamma \Rightarrow\ (C \sqcap D), \Delta}\ \sqcap\text{-r}$$

$$\frac{\Gamma, C \Rightarrow\ \Delta \qquad \Gamma, D \Rightarrow\ \Delta}{\Gamma, (C \sqcup D) \Rightarrow\ \Delta}\ \sqcup\text{-l} \qquad \frac{\Gamma \Rightarrow\ C, D, \Delta}{\Gamma \Rightarrow\ (C \sqcup D), \Delta}\ \sqcup\text{-r}$$

$$\frac{\Gamma, C \Rightarrow\ \Delta}{\Gamma, \forall R.C \Rightarrow\ \Delta}\ \forall\text{-l} \qquad \frac{\Gamma \Rightarrow\ C}{\Gamma \Rightarrow\ \forall R.C}\ \forall\text{-r}$$

$$\frac{\Gamma, C \Rightarrow\ \Delta}{\Gamma, \exists R.C \Rightarrow\ \Delta}\ \exists\text{-l} \qquad \frac{\Gamma \Rightarrow\ C, \Delta}{\Gamma \Rightarrow\ \exists R.C, \Delta}\ \exists\text{-r}$$

Figure 2: The System c$\mathcal{ALC}$: Sequent calculus

Readers not familiar with constructive reasoning please note that our version, which is constructive, has restrictions to a single conclusion formula in the rules for subsumption and universal-quantification-role on the right, which are essential to keep the system intuitionistic. It is reassuring to see the same rules for roles in Straccia's 4-valued Description Logic [Str97]. The rules for the propositional connectives ($\sqcap, \sqcup$) are basically the same as for classical ALC, and the rules for subsumption $\sqsubseteq$ are just the rules for intuitionistic implication.

The system $c\mathcal{ALC}$ [MS08] is related to constructive CK ([BPR01] and [MdP05]) in the same way classical multimodal K is related ALC[Sch91]. In the system $c\mathcal{ALC}$, the classical principles of the excluded middle $C \sqcup \neg C = \mathsf{T}$, double negation elimination $\neg\neg C = C$ and the definitions of the modalities $\exists R.C = \neg\forall R.\neg C$

and $\forall R.C = \neg\exists R.\neg C$ are no longer tautologies, but simply non-trivial TBox statements used to axiomatize specific application scenarios.

Soundness and completeness of a sequent calculus version of iALC is indicated in page 10 of [MS08], although not exactly for the sequent calculus we proposed in Figure 2. Our sequents are simpler than theirs, as we do not insist in carrying negative information along derivations, as they do. Nonetheless we have:

Theorem 2. *The sequent calculus for c$\mathcal{ALC}$ in Figure 2 and the Hilbert calculus described in Figure 1 are equivalent. For any TBox Θ and concept C, we have that $\Theta, \emptyset \vdash_H C$ if and only if the sequent $\Theta \Rightarrow C$ has a derivation using the rules in Figure 3.*

The proof of soundness and completeness of the sequent calculus for c$\mathcal{ALC}$ does not come straight from Straccia's work, as our rules for roles are the same, but our semantics are different. Straccia insists on 4-valuedness, we only want constructiveness.

Theorem 3. *The sequent calculus described in Figure 2 is sound and complete for TBox reasoning, that is $\Theta, \emptyset \models C$ if and only if $\Theta \vdash_S C$.*

3 Extending c$\mathcal{ALC}$

The main idea of the extension of **c**$\mathcal{ALC}$ with a constructive context operator $\Box$ from the logic CK [MdP05] is similar to the approach of Wolter and Zacharyaschev to modalizing classical description logics. Intuitively, we introduce a Kripke-style model to interpret the $\Box$ where each possible world is a **c**$\mathcal{ALC}$ model. We treat **c**$\mathcal{ALC}$ formulas as atomic formulas of the extended logic **c**$\mathcal{ALC}_\Box$.

If ϕ^{at} is a formula of **c**$\mathcal{ALC}$, then formulas of **c**$\mathcal{ALC}_\Box$ are defined as follows:

$$\phi ::= \top \mid \bot \mid \phi^{at} \mid \Box\phi \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \rightarrow \psi$$

The logic CK (see for example [MdP05]) is interpreted on models $(W, \leq, R, I)$ where W is a non-empty set of possible worlds, $\leq$ is a reflexive transitive binary relation on W, R is an arbitrary binary relation on W, and I is an interpretation function (p^I is a subset of W satisfying p). Inconsistent worlds are allowed, namely $\bot^I$ is not necessarily empty, so we have fallible worlds. The conditions on models are as follows:

- $\leq$ is hereditary with respect to atomic formulas, that is for every atomic p, if $w \in p^I$ and $w \leq w'$, then $w' \in p^I$ In particular, if $w \in \bot^I$ and $w \leq w'$, then $w' \in \bot^I$.

- if $w \in \bot^I$, then $w \in p^I$ for every atomic p.

A model of $\mathbf{c}\mathcal{ALC}_\Box$, $\mathcal{M}$, is a CK-model $(W, \leq, R, I)$ where in addition each W is a $\mathbf{c}\mathcal{ALC}$ model and for every formula ϕ^{at} of $\mathbf{c}\mathcal{ALC}$, $w \in (\phi^{at})^I$ iff $w \models \phi^{at}$.

Definition 1 (satisfaction in $\mathcal{M}$)**.** *The relation "the* $\mathbf{c}\mathcal{ALC}_\Box$ *-model* $\mathcal{M}$ *and the world* $w \in W$ *satisfy a formula* ϕ*" (in symbols* $\mathcal{M}, w \models \phi$*) is defined inductively as follows:*

$\mathcal{M}, w \models \phi^{at}$ *iff* $w \in (\phi^{at})^I$

$\mathcal{M}, w \models \phi \wedge \psi$ *iff* $\mathcal{M}, w \models \phi$ *and* $\mathcal{M}, w \models \psi$

$\mathcal{M}, w \models \phi \vee \psi$ *iff* $\mathcal{M}, w \models \phi$ *or* $\mathcal{M}, w \models \psi$

$\mathcal{M}, w \models \phi \rightarrow \psi$ *iff for all* w' *with* $w \leq w'$ *if* $\mathcal{M}, w' \models \phi$*, then* $\mathcal{M}, w' \models \psi$

$\mathcal{M}, w \models \Box\phi$ *iff for all* w' *with* $w \leq w'$ *,* $\forall u(R(w', u) \Rightarrow \mathcal{M}, u \models \phi)$

Theorem 4. *Satisfiability for* $\mathbf{c}\mathcal{ALC}_\Box$ *is decidable.*

Proof sketch. The decision procedure builds on decidability results for c$\mathcal{ALC}$ [MS08] and CK [MdP05], respectively. We formulate the procedure as a non-deterministic algorithm (guess a model and check that it is indeed a satisfying model) but it can also be described as a deterministic algorithm working by exhaustive enumeration of all models of fixed bounded size.

Given a formula ϕ of $\mathbf{c}\mathcal{ALC}_\Box$, we first use the result from [MS08] to guess a bounded size CK model M for ϕ (ignoring extra conditions for $\mathbf{c}\mathcal{ALC}_\Box$ models). Note that for each world of M, by construction from [MdP05], only the interpretation of subformulas of ϕ in M matters. Let us call this set of subformulas $Sf(\phi)$. Note that $Sf(\phi)$ is finite. Now we check, for each world w of M, whether the set $\{\phi^{at} : M, w \models \phi^{at}$ and $\phi^{at} \in Sf(\phi)\}$ has a $\mathbf{c}\mathcal{ALC}$ model. By the result of [MS08], this is decidable. If every world in M has a corresponding $\mathbf{c}\mathcal{ALC}$ model, we are done: we found a $\mathbf{c}\mathcal{ALC}_\Box$ model for ϕ. $\Box$

Having obtained decidability for $\mathbf{c}\mathcal{ALC}_\Box$ we conjecture that the system can be extended with many non-interacting boxes, to provide a system $c\mathcal{ALC}_{ctx}$ for Artificial Intelligence contexts, the application to Natural Language semantics that we are after.

4 Conclusions

We extended the constructive description logic c$\mathcal{ALC}$ with a modality box operator and proved that the resulting logic is decidable. This extension is motivated by a proposed application to modelling contexts in AI, as described, in the propositional setting, in [deP03]. Much remains to be done, in particular we want to investigate the complexity price of constructivity in our setting and we also must check the adaptation needs of our application.

References

[DL03] F. Baader, D. Calvanese, D. L. McGuinness, D. Nardi, and P. F. Patel-Schneider. *The description logic handbook: theory, implementation, and applications.* Cambridge University Press, 2003.

[BPR01] Bellin, G., de Paiva, V and E. Ritter. *Extended Curry-Howard Correspondence for a Basic Constructive Modal Logic*. In Procs of Methods for the Modalities 2, 2001.

[Boz09] Bozzato, L., M. Ferrari, P. Villa *A note on constructive semantics for description logics.* Accepted at CILC09 - 24-esimo Convegno Italiano di Logica Computazionale.

[BdP06] Braüner, T and de Paiva, V. *Intuitionistic hybrid logic.* J. Applied Logic (JAPLL) 4(3):231-255 (2006)

[HRP10] Haeusler, E. H., Rademaker, A. and de Paiva, V.*Using Intuitionistic Logic as a basis for Legal Ontologies* in 4th Workshop on Legal Ontologies and Artificial Intelligence Techniques, 2010, Firenze. CEUR Workshop Proceedings (CEUR-WS.org), 2010. v. 605. p. 69-76.

[deP03] de Paiva, V. *Constructive description logics: what, why and how.* Technical report, Xerox Parc, 2003.

[deP03a] de Paiva, V. *Natural Deduction and Context as (Constructive) Modality*. Modeling and Using Context, Proceedings of the 4th International and Interdisciplinary Conference CONTEXT 2003, Stanford, CA, USA, Springer Lecture Notes in Artificial Intelligence, vol 2680, June 2003.

[MdP05] Michael Mendler, Valeria de Paiva. *Constructive CK for Contexts* Procs. of the Worskhop on Context Representation and Reasoning, Paris, France, July 2005.

[MS08] Michael Mendler, Stephan Scheele. *Towards Constructive DL for Abstraction and Refinement.* Description Logics 2008.

[Neg05] Sara Negri. *Proof analysis in Modal Logic*

[RHP09] Rademaker, A., Haeusler, E. H. , Pereira, L. C. . *On the Proof Theory of ALC*. In: Walter Carnielli; Marcelo Coniglio; Itala D'Ottaviano. (Org.). The Many Sides of Logic. London: College Pulications, 2009, v. 21, p. 273-285.

[Ran09] Ranalter, K.*Embedding Constructive K into Intuitionistic K* in Electr. Notes Theor. Comput. Sci. 262: 205-219 (2010).

[Sch91] Schilds, K. *A correspondence theory for terminological logics: Preliminary report* PRoc. of the 12th IJCAI, Sydney Australia, 1991.

[Sim95] Simpson, A.*The Proof Theory and Semantics of Intuitionistic Modal Logic*. PhD Thesis, University of Edinburgh, December 1993, revised September 1994.

[Str97] Straccia, U.*A sequent calculus for reasoning in four-valued description logics* TABLEAUX 1997. LNAI 1227, Pont-a-Mousson, France, 1997.

[WZ99] Wolter, F. and M. Zakharyaschev *Modal Description Logics: modalizing roles*. Fundamenta Informaticae. 39 (1999), pp.411-438.